Strong Shrinkage of the Effective Core at High Energy*

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On the basis of analyticity alone it is shown that in the high-energy limit the partial-wave amplitude $a_l(s)$ for $l>$const. $\ln^2 s$ do not contribute substantially to the scattering amplitude for finite angles. For an angle depending upon s this result can be extended to show that the contribution to the scattering amplitude $T[s, \cos\theta(s)]$ from $a_l(s)$ for $l>$const. $\ln^2 s/\sin\theta(s)$ is smaller than any power of s. Here s and θ are the square of the energy and the scattering angle in the center-of-mass system. Classically, this means that a bombarding particle whose impact parameter is larger than the shrinking quantity const. $\ln^2 s/s^{1/2}\sin\theta(s)$ cannot be appreciably scattered through an angle $\theta(s)$. It is necessary to use the unitarity condition only when this result is used to put an upper bound on $T[s, \cos\theta(s)]$. Here $T(s, \cos\theta)$ is assumed to be analytic as a function of $\cos\theta$ in an s-independent complex neighborhood of the real segment $(-1, 1)$ except for its intersection with the cuts from ∞ to $x(s)$ and from $-x(s)$ to $-\infty$, $x(s)$ being an arbitrary function with $x(s)>1$. Finally the forward and backward peaks are discussed and it is proved that the right- (left-) hand cut contributes only to the forward (backward) peak and not to the backward (forward) peak.

I. INTRODUCTION AND STATEMENT OF RESULTS

IT is well known that analyticity and unitarity impose rather strong restriction on the high-energy behavior of scattering amplitude. Recently, Kinoshita, Loeffel, and Martin[1] have found that the scattering amplitude $T(s, \cos\theta)$ for scalar particles satisfies the inequality

$$|T(s, \cos\theta)| < \text{const. } \ln^{3/2} s \tag{1}$$

for sufficiently large s when $\cos\theta \neq \pm 1$. Here s and θ are the square of the center-of-mass energy and the center-of-mass scattering angle, respectively. This inequality has great interest since it implies that the differential cross section for finite angle scattering must be bounded by

$$d\sigma/d\Omega < \text{const. } \ln^3 s/s\,, \tag{2}$$

which decreases rapidly as s increases. The inequality (1) holds because, for large angular momentum l, the partial-wave amplitude $a_l(s)$ is a slowly varying function of l and therefore, for finite angles, the contributions from successive partial waves tend to cancel. It seems reasonable that for a finite angle θ_0 the remaining contribution from $a_l(s)$ to $T(s, \cos\theta_0)$ comes only from $a_l(s)$ with small l.

Before treating the general proof we shall give a simple nonrigorous derivation of the result. We assume[2] that $T(s, \cos\theta)$ is analytic in an ellipse in the complex $\cos\theta$ with foci at $\pm\cos\theta_0$ and with a semimajor axis of length 1 and that $T(s, \cos\theta)$ is bounded by s^{N_1} on this ellipse, where N_1 is some positive number. Now consider the following Legendre expansion (we put $\cos\theta = z$).

$$T(s,z) = \sum_{l=0}^{\infty} (2l+1) b_l(s) P_l(z/\cos\theta_0)\,, \tag{3}$$

where

$$b_l(s) = \frac{1}{2}\int_{-1}^{1} dz P_l(z) T(s, z\cos\theta_0)\,. \tag{4}$$

From the analyticity of $T(s,z)$ in the ellipse, we have[3]

$$|b_l(s)| < \text{const.} s^{N_1} |\sec\theta_0 + \tan\theta_0|^{-l}. \tag{5}$$

We shall decompose $T(s,z)$ into two parts $T'(s,z)$ and $T(s,z) - T'(s,z)$ with

$$T'(s,z) = \sum_{l=0}^{l_0} (2l+1) b_l(s) P_l(z/\cos\theta_0)\,. \tag{6}$$

From the inequality (5) we see that for some N such that

$$l_0 > (N+N_1) \ln s/\ln|\sec\theta_0 + \tan\theta_0|\,, \tag{7}$$

$T(s,z) - T'(s,z)$ is smaller than s^{-N} for $\cos\theta_0 \geqslant z \geqslant -\cos\theta_0$. Therefore, by choosing l_0 and N sufficiently large, we can say that only $T'(s,z)$ contributes to $T(s,z)$ in $\cos\theta_0 \geqslant z \geqslant -\cos\theta_0$. Even when $T'(s,z)$ is re-expressed in the usual partial-wave expansion, coefficients for $l > l_0$ will still be zero since $T'(s,z)$ is just a polynomial of order l_0. Thus, the scattering amplitude for a finite angle can be expressed to arbitrary accuracy with the partial waves whose angular momenta l are smaller than $(N+N_1) \ln s/\ln|\sec\theta_0 + \tan\theta_0|$. Thus it will be expected that for finite angle scattering the amplitude has no contribution from the partial waves for angular momenta larger than $(N+N_1) \ln s/\ln|\sec\theta_0 + \tan\theta_0|$. However, the above discussion is not conclusive because we cannot prove that the coefficients of the usual partial-wave expansion of $T(s,z) - T'(s,z)$, for angular momenta smaller than $(N+N_1) \ln s/\ln|\sec\theta_0 + \tan\theta_0|$ are equal to zero. At this stage it should be noted that we have not used unitarity.

The purpose of this paper is to prove the above stated expectation. Before entering upon the details of the proof, we want to discuss the procedure. It would be

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[1] T. Kinoshita, J. J. Loeffel, and A. Martin, Phys. Rev. Letters **10**, 460 (1963), hereafter referred to as KLM.
[2] This assumption is weaker than that of KLM.
[3] O. W. Greenberg and F. E. Low, Phys. Rev. **124**, 2047 (1961).

sufficient to show that we can choose constant c in such a way that $|T''(s,z)|<s^{-N}$ for sufficiently large N, where $T''(s,z)$ is

$$T''(s,z)=\sum_{l=c\ \ln s}^{\infty}(2l+1)a_l(s)P_l(z). \tag{8}$$

Such a proof is impossible,[4] however, since we do not expect that $a_l(s)$ for $l>c\ \ln s$ is equal to zero. But for a finite angle scattering amplitude, we expect the contribution from $a_l(s)$ for $l>c\ \ln s$ to cancel each other. The discontinuity in the division in the definition of $T''(s,z)$ at a certain l will have the result that the lowest terms will not be cancelled, no matter how large the constant c. This is just a mathematical difficulty, therefore we make a less abrupt division by introducing a slowly varying function $f_l(s)$ in the division. $f_l(s)$ is characterized by the properties:

(1) $f_l(s)$ is a monotonically decreasing function of l,

(2) $|1-f_l(s)|<s^{-N_2}$ for $l<l_1$, (9)

(3) $|f_l(s)|<s^{-N_3}$ for $l>l_2$, (10)

where N_2 and N_3 are arbitrarily large numbers. Using $f_l(s)$ the summation for $T(s,z)$ is divided into an upper part

$$T_u(s,z)=\sum_{l=0}^{\infty}(2l+1)a_l(s)[1-f_l(s)]P_l(z) \tag{11}$$

and a lower part $T(s,z)-T_u(s,z)$. We now have to show

$$|T_u(s,z)|<s^{-N} \tag{12}$$

for $\cos\theta_0\geqslant z\geqslant-\cos\theta_0$ and arbitrarily large N.

It will be convenient to assume a particular $f_l(s)$ with the above properties:

$$f_l(s)=\left[1-\exp\left(-\frac{\alpha\ \ln^2 s}{l+1}\right)\right]^{\beta\ \ln s}, \tag{13}$$

where α and β are some positive adjustable parameters. We can easily see that $f_l(s)$ satisfies the property (2) and (3) with

$$l_1=m\ \ln s \quad\text{and}\quad l_2=m'\ \ln^2 s, \tag{14}$$

provided that

$$\alpha>mN_2, \tag{15}$$

and

$$-\beta\ \ln[1-\exp(-\alpha/m')]>N_3. \tag{16}$$

The proof of the inequality (12) is now our main task. Another choice of $f_l(s)$ might result in a smaller l_2 than Eq. (14) for the same N, N_2, and N_3. But, at least, we can conclude that partial waves with[5] $l>m'\ \ln^2 s$ do not contribute to finite angle scattering. Classically this result means that bombarding particles whose impact parameters are larger than $m'\ \ln^2 s/s^{1/2}$ cannot be scattered by finite angles but only into the forward and backward peaks.

For the proof of the inequality (12) we shall make use of the analyticity condition alone. In Sec. II, we shall accomplish the proof of the inequality (12) by requiring as the analyticity condition full analyticity deduced from the Mandelstam representation. However, as will be shown in Sec. III, a proof with weaker analyticity assumption can be given. A sufficient assumption is that $T(s,z)$ be analytic in a domain D. D is an s-independent complex neighborhood of the real segment $(-1,1)$ except for its intersection with the cuts from ∞ to $x(s)$ and $-x(s)$ to $-\infty$, $x(s)$ being an arbitrary function with $x(s)>1$.

The discussions in Secs. II and III can also be applied to the amplitude at an angle $\theta(s)$ which is dependent on s. This application will be made in Sec. IV. The only change in Eqs. (13) and (14) will be to replace l by $l\sin\theta(s)$. The result is that $T[s,\cos\theta(s)]$ has no significant contribution from $a_l(s)$ for $l>$const. $\ln^2 s/\sin\theta(s)$.

Throughout this paper, we do not use the unitarity condition. However, in order to fix an upper bound of $T[s,\cos\theta(s)]$ using the results of this paper, it would be necessary to use the unitarity bound for the $a_l(s)$ with $l<$const. $\ln^2 s/\sin\theta(s)$.

Finally in Sec. V we briefly discuss the forward and backward peaks and it will be proved that the right- (left-) hand cut contributes only to the forward (backward) peak and not to the backward (forward) peak.

Although we only discuss the scattering amplitude for two scalar particles, it is easily seen that these considerations can be generalized for the scattering of particles with spin by using the technique introduced in a previous note.[6]

II. PROOF BASED ON THE MANDELSTAM REPRESENTATION

In this section we shall prove the inequality (12) on the basis of the Mandelstam representation. The logic of proof is quite similar to that of KLM. According to KLM, we consider the function

$$G(s,z)=\sum_{l=0}^{\infty}(2l+1)a_l(s)z^l. \tag{17}$$

We have the following relations between $T(s,z)$ and

[4] We cannot apply the proof in Sec. II to $T''(s,z)$ since the M_3 for $G_u''(s,z)$ of $T''(s,z)$ depends on the constant c and because of this dependence we cannot make the M_2 for $G_u''(s,z)$ small.

[5] The reason for the appearance of the factor $\ln^2 s$ instead of $\ln s$ which might seem more reasonable is the very severe condition in the inequality (10). The inequality (9) is necessary for the proof of the inequality (12), but the inequality (10) is not. If we would content ourselves with a weaker condition for the inequality (10) the $\ln^2 s$ dependence of l_2 might be replaced by smaller power of $\ln s$ by using another function as $f_l(s)$. $f_l(s)$ in Eq. (13) may not be the best one for our purposes but is only an example. Therefore, $\ln^2 s$ might possibly be replaced by a smaller value without weakening the inequality (10) by taking another $f_l(s)$.

[6] K. Yamamoto, Nuovo Cimento 27, 1277 (1963).

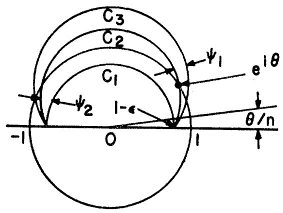

FIG. 1. Circular arcs C_1, C_2, and C_3.

$G(s,z)$:

$$T(s,z)=\frac{1}{\pi}\int_0^\pi G[s,z+(z^2-1)^{1/2}\cos t]dt, \tag{18}$$

$$=\frac{1}{\pi i}\int_{z-(z^2-1)^{1/2}}^{z+(z^2-1)^{1/2}}\frac{G(s,z')dz'}{(1-2zz'+z'^2)^{1/2}}, \tag{19}$$

$$G(s,z)=\int_{-1}^{1}\frac{(1-z^2)T(s,z')dz'}{(1-2zz'+z^2)^{3/2}}. \tag{20}$$

We assume that $T(s,z)$ is analytic in the z plane except for cuts from ∞ to $x(s)$ and $-x(s)$ to $-\infty$, $x(s)$ being an arbitrary function with $x(s)>1$. Then $G(s,z)$ is analytic in the z plane except for cuts from ∞ to $x+(x^2-1)^{1/2}$ and $-x-(x^2-1)^{1/2}$ to $-\infty$.

The following theorem[7] quoted by KLM is useful for our discussion. In the upper half complex z plane, draw the following three circular arcs C_1, C_2, and C_3 cutting the real axis at $\pm(1-\epsilon)$ (see Fig. 1). C_1 is an arc centered at the origin. C_2 is an arc passing through the point $z=e^{i\theta_0}$. C_3 is an arc outside C_2 and has at least a finite angle of intersection φ_1 with C_2. The intersection angle φ_2 between C_1 and C_2 is

$$\tan\varphi_2=\frac{\epsilon(1-\epsilon/2)}{\sin\theta_0(1-\epsilon)}. \tag{21}$$

Now let $\psi(z)$ be an analytic function defined in a domain bounded by the two arcs C_1 and C_3. Then the upper bound of $|\psi(z)|$ on C_2 is

$$M_2=M_1{}^{\varphi_1/(\varphi_1+\varphi_2)}M_2{}^{\varphi_2/(\varphi_1+\varphi_2)}, \tag{22}$$

where M_i is the upper bound of $|\psi(z)|$ on C_i. Applying this theorem to $G(s,z)$, KLM have obtained the inequality (1).

We shall apply the above theorem to the function defined by

$$G_u(s,z)=\sum_{l=0}^{\infty}(2l+1)a_l(s)[1-f_l(s)]z^l. \tag{23}$$

Between $G_u(s,z)$ and $T_u(s,z)$, there is the relation

$$T_u(s,z)=\frac{1}{\pi i}\int_{z-(z^2-1)^{1/2}}^{z+(z^2-1)^{1/2}}\frac{G_u(s,z')dz'}{(1-2zz'+z'^2)}. \tag{24}$$

[7] Instead of this theorem, we can use Hadamard's three-circles theorem [see, for example, J. E. Littlewood, *Theory of Functions* (Oxford University Press, London, 1944), p. 113], as was used by T. Kinoshita for the derivation of the inequality (1). T. Kinoshita, Lecture Note at the Internationale Universitätswoche für Kernphysik at Schladming, Austria, 1963 (unpublished).

As will be proved in Appendix A, $G_u(s,z)$ is analytic in the domain limited by C_1 and C_3 provided that ϵ and $1-\cos\theta_0$ are finite, and M_3 for $G_u(s,z)$ is

$$M_3=\text{const.}s^{\beta\ \ln 2+N_4}, \tag{25}$$

where const. and N_4 are finite numbers which depend only on $T(s,z)$. M_1 for $G_u(s,z)$ will be estimated in Appendix B and the result is

$$M_1=\text{const. }(\ln^3 s)s^{-[\alpha\ \ln(1-\epsilon)]^{1/2}+N_5} \tag{26}$$

for a sufficiently large N_5 which depends only on $T(s,z)$. Therefore, apart from unimportant factors

$$\ln M_2\sim\left\{(-[-\alpha\ln(1-\epsilon)]^{1/2}+N_5)\frac{\varphi_1}{\varphi_1+\varphi_2}+(\beta\ln 2+N_4)\frac{\varphi_2}{\varphi_1+\varphi_2}\right\}\ln s. \tag{27}$$

From Eq. (24) we have, for $\cos\theta_0\geqslant\cos\theta\geqslant-\cos\theta_0$,

$$|T_u(s,\cos\theta)|<\frac{2}{\pi}\int_0^1\frac{G_u(s,re^{i\theta})dr}{(1-r)^{1/2}}<\frac{4}{\pi}M_2, \tag{28}$$

since M_2 is also the maximum of $G_u(s,z)$ between C_1 and C_2. Thus the inequality (12) becomes

$$([-\alpha\ln(1-\epsilon)]^{1/2}-N_5)\frac{\varphi_1}{\varphi_1+\varphi_2}-(\beta\ln 2+N_4)\frac{\varphi_2}{\varphi_1+\varphi_2}>N. \tag{29}$$

For fixed value of N, N_i, m, $\cos\theta_0$, ϵ, φ_i and β, we can take α large enough to satisfy the inequalities (15) and (29). Next we can take m' large enough to satisfy the inequality (16).

The above discussion is sufficient to prove the existence of a function $f_l(s)$ having the four properties (1), (2), (3), and the property described by the inequality (12).

III. PROOF BASED ON ANALYTICITY IN D

In this section we shall show that for proof of the inequality (12) it is sufficient to assume analyticity only in the domain D which was defined at the end of Sec. I. Using the Cauchy integral we shall divide $T(s,z)$ in two parts $T_1(s,z)$ and $T_2(s,z)$ where

$$T_1(s,z)=\frac{1}{2\pi i}\int_{C_4}\frac{T(s,z')dz'}{z'-z}, \tag{30}$$

and

$$T_2(s,t)=\frac{1}{2\pi i}\int_{C_5}\frac{T(sz')dz'}{z'-z}, \tag{31}$$

where C_4 plus C_5 form the boundary of D and C_5 is the part of the boundary along the cuts. This division is useful because $T_2(s,z)$ is analytic except for the short cuts. Consider the partial-wave expansion of $T_1(s,z)$ and $T_2(s,z)$:

$$T_1(s,z)=\sum_{l=0}^{\infty}(2l+1)a_l'(s)P_l(z), \tag{32}$$

and

$$T_2(s,z)=\sum_{l=0}^{\infty}(2l+1)a_l''(s)P_l(z). \tag{33}$$

From the definition (30), we have[3]

$$|a_l'(s)|<\text{const.}s^{N_6}[a+(a^2-1)^{1/2}]^{-l}, \tag{34}$$

where a is the length of the semimajor axis of the largest ellipse with foci at $z=\pm1$ which we can draw without intersecting C_4. We have also assumed that $T(s,z)$ is bounded by s^{N_6} on C_4.

Consider $T_{1u}(s,z)$ and $T_{2u}(s,z)$ defined by

$$T_{iu}(s,z)=\sum_{l=0}^{\infty}(2l+1)a_l^{(i)}(s)[1-f_l(s)]P_l(z). \tag{35}$$

Then

$$T_u(s,z)=T_{1u}(s,z)+T_{2u}(s,z). \tag{36}$$

We shall obtain an upper bound on $T_u(s,z)$ by treating $T_{1u}(s,z)$ and $T_{2u}(s,z)$ separately. First, using the inequality (34) and Eq. (35) we obtain an upper bound of $T_{1u}(s,z)$ in $\cos\theta_0\geqslant z\geqslant -\cos\theta_0$:

$$\begin{aligned}|T_{1u}(s,z)|&<\text{const.}s^{N_6}[1-f_{x\ln s-1}(s)]\sum_{l=0}^{x\ln s-1}(2l+1)\\ &\times[a+(a^2-1)^{1/2}]^{-l}+\text{const.}s^{N_6}\sum_{l=0}^{\infty}(2l+2x\ln s+1)\\ &\times[a+(a^2-1)^{1/2}]^{-l-x\ln s}<\text{const.}(\ln s)s^{N_6-\alpha/x}\\ &+\text{const.}(\ln s)s^{N_6}[a+(a^2-1)^{1/2}]^{-x\ln s}\\ &+\text{const.}s^{N_6}[a+(a^2-1)^{1/2}]^{-x\ln s}.\end{aligned} \tag{37}$$

Putting $x=\{\alpha/\ln[a+(a^2-1)^{1/2}]\}^{1/2}$, we have

$$|T_{1u}(s,z)|<\text{const.}(\ln s)s^{N_6-\{\alpha/\ln[a+(a^2-1)^{1/2}]\}^{1/2}}, \tag{38}$$

where the constant coefficients are finite so long as $a-1$ and α are finite.

Next we investigate the upper bound of $|T_{2u}(s,z)|$ in $\cos\theta_0\geqslant z\geqslant -\cos\theta_0$. From definition (31), $T(s,z)$ is analytic except for short cuts near $z=\pm1$. We can apply the results of the Sec. II here, since in Sec. II we have assumed only the analyticity of $T(s,z)$ except for cuts which in that case reached to infinity and did not use the unitarity condition. Therefore, it is obvious that by choosing α sufficiently large we can make

$$|T_{2u}(s,z)|<s^{-N} \tag{39}$$

in $\cos\theta_0\geqslant z\geqslant -\cos\theta_0$. From Eq. (36) and the inequalities (38) and (39), it may be seen that the sufficient condition for inequality (12) is the inequality

$$[\alpha/\ln(a+(a^2-1)^{1/2})]^{1/2}-N_6>N. \tag{40}$$

This inequality will hold for sufficiently large α.

IV. SMALL ANGLE SCATTERING

The discussions in the two previous sections for finite angle scattering can also be applied to the scattering amplitude at angles dependent upon s. Results of interest may be obtained for angles converging with energy to 0 or π. For the extension we define a new $f_l(s)$ as

$$f_l(s)=\left[1-\exp\frac{-\alpha\ln^2 s}{(l+1)\sin\theta(s)}\right]^{\beta\ln s} \tag{41}$$

instead of (13). l_1 and l_2 in the inequalities (9), (10) and Eq. (14), therefore, change into

$$l_1=m\ln s/\sin\theta(s), \tag{42}$$

and

$$l_2=m'\ln^2 s/\sin\theta(s). \tag{43}$$

The proof of the inequality (12) is quite similar to the previous one. The main changes are: (a) θ_0 is not finite but proportional to $\sin\theta(s)$, (b) ϵ is not finite but proportional to $\sin\theta(s)$ so that φ_1 and φ_2 are finite, and (c) α in Secs. II, III, and Appendices A, B must be replaced by $\alpha/\sin\theta(s)$. There are also other small changes. For example some coefficients in Secs. II, III, and Appendices A, B go to infinity as s increases. These changes do not affect the proof of the inequality (12), however, since they can be overcome by choosing α and m' sufficiently large. Thus we obtain the result that partial waves for l larger than const. $\ln^2 s/\sin\theta(s)$ do not contribute to $T[s,\cos\theta(s)]$.

V. FORWARD AND BACKWARD PEAKS

Finally we shall briefly discuss the forward and the backward peaks and prove that the right- (left-) hand cut contributes only to the forward (backward) peak and not to the backward (forward) peak. We shall show this only for the left-hand cut, since exactly the same considerations hold for the right-hand cut. We shall denote the integration along the left-hand cut in Eq. (31) by $T_2^L(s,z)$. As is shown in Appendix C, if $a_l^L(s)$ is the partial-wave amplitude of $T_2^L(s,z)$,

$$G_{2u}^L(s,z)=\sum_{l=0}^{\infty}a_l^L(s)[1-f_l(s)]z^l \tag{44}$$

is analytic in the domain in which $|z|$ is finite and $\pi-\theta_0/n>\arg z>-\pi+\theta_0/n$ for sufficiently large n.

Now consider the three circular arcs C_1', C_2', and C_3' in the right-half complex plane obtained by rotating C_1, C_2, and C_3 of Sec. II through $\frac{1}{2}\pi$ about the origin in the clockwise direction. We can then apply the theorem used in Sec. II to $G_{2u}^L(s,z)$, since as will be shown in Appendix C the maxima M_1^L, M_3^L of $G_{2u}^L(s,z)$ on C_1' and C_3' can be expressed in a way similar to those for $G_u(s,z)$ on C_1 and C_3. Thus we can conclude that for $|z|\leqslant 1$ and $\pi-\theta_0/n>\arg z>-\pi+\theta_0/n$

$$|G_{2u}{}^L(s,z)|<s^{-N} \tag{45}$$

for an arbitrary N if α and m' are sufficiently large. However, the relation between $G_{2u}{}^L(s,z)$ and

$$T_{2u}{}^L(s,z)=\sum_{l=0}^{\infty}(2l+1)a_l{}^L(s)[1-f_l(s)]P_l(z) \tag{46}$$

is

$$T_{2u}{}^L(s,z)=\int_0^{\pi} G_{2u}{}^L[s,\, z+(z^2-1)^{1/2}\cos t]dt. \tag{47}$$

Therefore, $T_{2u}{}^L(s,z)$ has no forward peak. This means that $T_2{}^L(s,z)$ has no forward peak because the contribution to this peak must come only from the partial-wave amplitudes in the upper sum. The reason is that the forward amplitude increases more rapidly than the amplitude in the lower summation is compelled to by partial-wave unitarity. If there is a backward peak, the same argument holds.

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APPENDIX A: ANALYTICITY AND UPPER BOUND FOR $G_u(s,z)$ ON C_3

In order to study the analyticity of $G_u(s,z)$ in the domain limited by C_1, C_3 and the upper bound for $G_u(s,z)$ on C_3, we need only to discuss the domain $\pi-\theta_0/n>\arg z>\theta_0/n$, where n is a large positive number. This is true because for sufficiently large n outside $\pi-\theta_0/n>\arg z>\theta_0/n$, $|z|<1$ (see Fig. 1). For $|z|<1$, the analyticity of $G_u(s,z)$ is obvious because the power-series expansion converges explicitly for any s because, by analyticity, the $a_l(s)$ are bounded by

$$|a_l(s)|<s^{N_5} \tag{48}$$

for some sufficiently large N_5. To prove the analyticity and bound of $G_u(s,z)$ in $\pi-\theta_0/n>\arg z>\theta_0/n$, it is necessary to use analyticity of $a_l(s)$. The n' times subtracted dispersion relation for $T(s,z)$ at fixed s is

$$T(s,z)=\frac{1}{\pi}z^{n'}\int_{x(s)}^{\infty}dz'\frac{\rho(s,z')}{z'^{n'}(z'-z)} +\frac{1}{\pi}z^{n'}\int_{x(s)}^{\infty}dz'\frac{\rho'(s,z')}{z'^{n'}(z'+z)}+\frac{1}{\pi}\sum_{p=1}^{n'-1}\rho_p Z^p. \tag{49}$$

From Eq. (49), we have for $l\geqslant n'$ [8]

$$a_l(s)=b_l(s)+(-)^l b_l'(s), \tag{50}$$

where

$$b_l(s)=\frac{1}{\pi}\int_{x(s)+[x^2(s)-1]^{1/2}}^{\infty}d\zeta\frac{\sigma(s,\zeta)}{\zeta^{l+1}}, \tag{51}$$

with

$$\sigma(s,\zeta)=\int_{x(s)}^{(\zeta+1/\zeta)}dz'\frac{\rho(s,z')}{(1-2\zeta z'+\zeta^2)^{1/2}}. \tag{52}$$

This is true for $b_l'(s)$ also. From Eq. (50) we see that $b_l(s)$ and $b_l'(s)$ are analytic in the complex l plane for $\mathrm{Re}\,l>n'$ for all physical values of s. Using this analyticity, we can rewrite $G_u(s,z)$ for $\pi-\theta_0/n>\arg z>\theta_0/n$ as

$$G_u(s,z)=P(s,z)+I(s,z), \tag{53}$$

where

$$P(s,z)=\sum_{l=0}^{n'}(2l+1)a_l(s)[1-f_l(s)]z^l, \tag{54}$$

and

$$I(s,z)=\frac{i}{2}\int_{n'+\frac{1}{2}-i\infty}^{n'+\frac{1}{2}+i\infty}dl \times\frac{(2l+1)[1-f_l(s)][b_l(s)(e^{-i\pi}z)^l+b_l'(s)z^l]}{\sin\pi l}. \tag{55}$$

In Eq. (55) we have used the Watson-Sommerfeld transformation. We shall only study the analyticity and the upper bound of $I(s,z)$ since these properties are obvious for $P(s,z)$. From Eq. (55) it is clear that $I(s,z)$ is analytic in the domain in which $|z|$ is finite and $\pi-\theta_0/n>\arg z>\theta_0/n$.

We estimate the upper bound of $I(s,z)$ on C_3 by using Eq. (55). For arbitrary l satisfying $l=n'+\frac{1}{2}+i\eta$, we have from Eq. (13)

$$|1-f_l(s)|<2^{\beta\ \ln s}, \tag{56}$$

and we also have from Eq. (51)[9]

$$\mathrm{Max}[|b_l(s)|,|b_l'(s)|]<|l|^{n'}s^{N_4} \tag{57}$$

for sufficiently large N_4. Therefore, for z on C_3 in $\pi-\theta_0/n>\arg z>\theta_0/n$ and for $l=n'+\frac{1}{2}+i\eta$

$$|b_l(s)(e^{-i\pi}z)^l+b_l'(s)z^l| <|l|^{n'}s^{N_4}|z|^{n'+\frac{1}{2}}\exp(\pi\eta-\theta_0\eta/n). \tag{58}$$

Substituting the inequalities (56), (57), and (58) in Eq. (55), we have

$$|I(s,z)|<c|z|^{n'+1/2}s^{\beta\ \ln 2+N_4}, \tag{59}$$

with

$$c=4\int_0^{\infty}d\eta|n'+1+i\eta||n'+\tfrac{1}{2}+i\eta|^{n'}\exp(-\theta_0\eta/n). \tag{60}$$

[8] M. Froissart, Phys. Rev. **123**, 1053 (1961).

[9] Practically, as we can see from the discussion in Sec. III, it is sufficient to consider the case where $\rho(s,z)$ in Eqs. (49) and (52) is zero except for a finite range of z.

APPENDIX B: UPPER BOUND FOR $G_u(s,z)$ IN $|z|<1$

In this Appendix we estimate the upper bound of $G_u(s,z)$ in $|z|<1$. Using the inequality (48) and Eq. (23), for $|z|<1$, we have

$$|G_u(s,z)|<s^{N_5}[1-f_{x\ln s-1}(s)]\sum_{l=0}^{x\ln s-1}(2l+1)$$
$$+s^{N_5}\sum_{l=0}^{\infty}(2l+2x\ln s+1)|z|^{l+x\ln s}, \quad (61)$$
$$<\text{const.}(\ln^3 s)s^{N_5-\alpha/x}+\text{const.}|z|^{x\ln s}s^{N_5}$$
$$+\text{const.}z^{x\ln s}s^{N_5}\ln s. \quad (62)$$

Putting $x=[-\alpha/\ln|z|]^{1/2}$ and $|z|=1-\epsilon$, we obtain the inequality (26).

APPENDIX C: ANALYTICITY AND THE UPPER BOUND FOR $G_{2u}{}^L(s,z)$

The difference between $G_{2u}{}^L(s,z)$ and $G_{2u}(s,z)$ essential for this argument is the absence of the right-hand cut in $G_{2u}{}^L(s,z)$. By virtue of this difference, we have only $b_l'(s)$ and no term corresponding to $b_l(s)$ in Eq. (55). Therefore it is obvious that $G_{2u}{}^L(s,z)$ is analytic in the domain in which $|z|$ is finite and $\pi-\theta_0/n>\arg z>-\pi-\theta_0/n$. By the procedure used for the calculation of the bounds for $G_u(s,z)$ in Appendices A and B, we can obtain similar upper bounds for $G_{2u}{}^L(s,z)$ on C_1' and C_3'.

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Overlapping Resonances in Dispersion Theory*

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The Khuri-Treiman dispersion representation is applied to the discussion of overlapping resonances among particles in production and decay final states. The kernel of the dynamical equation following from the Khuri-Treiman representation has branch points overlapping the integration contour, but recently reported work permits us to select the correct branch of the kernel. We thus eliminate all restrictions on the masses of the final-state particles or strengths of the resonances. An iteration procedure is developed for the solution of the dynamical equation when three spinless particles are present in the final state. There is no restriction on the angular momentum of the resonances, but for simplicity only s-wave resonances are considered here. Plausibility arguments are given which indicate that for narrow resonances the once-iterated approximation to the solution is a good approximation. A detailed study of all higher approximations supports this assertion. In the once-iterated approximation, one finds a branch point on the second sheet of the transition amplitude which may cause a characteristic variation of the amplitude near the low-energy boundaries of the physical region. This variation is studied quantitatively for the kinematically favorable reaction $N+N\to N+N+\pi$, and is found to be of negligible importance. The suppression of the variation is related to the threshold behavior of two-particle scattering amplitudes.

I. INTRODUCTION

In this paper we discuss the role of resonant final-state interactions in production and decay reactions leading to three-particle final states. In particular, we study what happens when two of the three outgoing particles are identical and either one (or both) scatters resonantly with the third. Following Peierls and Tarski,[1] we call this the case of overlapping resonances. As is well known, this class of reactions includes cases of great current interest, for instance,

$$\bar{K}+N\to Y^*+\pi\to\begin{pmatrix}\Lambda+\pi+\pi\\ \Sigma+\pi+\pi\end{pmatrix},$$

$$\pi+N\to N^*+\pi\to N+\pi+\pi,$$
$$N+N\to N^*+N\to N+N+\pi,$$

where the π-π resonances are excluded kinematically. The restrictions to two identical particles and only two resonances are made for convenience. The methods we use can be extended to study π-π resonances in the reactions above, or to study

$$\bar{K}+N\to\begin{pmatrix}\bar{K}^*+N\\ N^*+\bar{K}\end{pmatrix}\to N+\bar{K}+\pi,$$

$$\bar{N}+N\to\begin{pmatrix}\rho+\pi\\ f^0+\pi\end{pmatrix}\to\pi+\pi+\pi.$$

The dynamics of our treatment are provided by the Khuri-Treiman[2] (KT) dispersion representation of a

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[1] R. F. Peierls and J. Tarski, Phys. Rev. **129**, 981 (1963).
[2] N. N. Khuri and S. B. Treiman, Phys. Rev. **119**, 1115 (1960).